Lecture 13 **2012/12/24**

Outline

1. **Motivation**
2. **LaSalle-Krosovskii's Theorem**
3. **Region of Attraction**
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1. Motivation

- LaSalle-Krosovskii's Theorem makes some extensions of Lyapunov theorems for AS. It has an insight link with limit sets, which are very important concepts in qualitative analysis of ODE.
- In real applications, finding the estimation of a region of attraction is necessary. Lyapunov function plays an important role doing so.

2. LaSalle-Krosovskii's Theorem

1) A Motivated Example

Consider the pendulum equation with friction given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases}. \tag{13.1}$$

We have that $V(x) = \dfrac{g}{l}(1-\cos x_1) + \dfrac{1}{2}x_2^2 > 0\ (-2\pi < x_1 < 2\pi) \quad \Rightarrow \quad \dot{V}(x) = -\dfrac{k}{m}x_2^2$ negative semi-definite.

We see that $\dot{V}(x) < 0$ except for $x_2 = 0$, where $\dot{V}(x) = 0$. For the system (13.1) maintaining $\dot{V}(x) = 0$, the trajectory of (13.1) must be confined to $x_2 = 0$. Unless $x_1 = 0$, it is impossible from the pendulum equation with friction

$$x_2(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_2(t) \equiv 0 \quad \Rightarrow \quad \sin x_1(t) \equiv 0 \quad \Rightarrow \quad x_1(t) \equiv 0.$$

Therefore, $V(x(t))$ must decrease toward to zero and, consequently, $\lim_{t\to\infty} x(t) = 0$ because $V(x)$ is positive definite. This is consistent with the fact that, due to friction, energy can't remain constant while the system is in motion.

Remark 13.1 If we assume that no trajectories can stay identically at points where

$\dot{V}(x) = 0$, except at the origin, the origin is AS in the case of $V(x) > 0$ and $\dot{V}(x) \le 0$. This is a basic idea of LaShalle-Krosovskii's Theorem, which links with concepts of limit sets.

2) Three Fundamental Properties of Dynamic System

Consider an autonomous system

$$\dot{x} = f(x), \tag{13.2}$$

where $f : D \to R^n$ is locally Lip., $D \subseteq R^n$ and $f(0) = 0$. Assume that (13.2) is complete for each $x_0 \in R^n$.

Definition 13.1 Let $x(t; t_0, x_0)$ be a solution of (13.2), passing through the point (t_0, x_0). $\{x(t; t_0, x_0) \mid t \in R\} \subseteq R^n$ is called an **orbit** or **trajectory** of (13.2), which is also denoted as $\gamma(x_0)$. We denoted by $\gamma^+(x_0)$ **a positive orbit** defined by $x(t; t_0, x_0)$, $t \ge 0$; by $\gamma^-(x_0)$ **a negative orbit** defined by $x(t; t_0, x_0)$, $t \le 0$. Then, $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$. If $x(t; t_0, x_0)$ is periodic, $\gamma^+(x_0) = \gamma^-(x_0)$.

Remark 13.1 Look at the difference between a solution $x(t; t_0, x_0) \in R \times R^n$ of (13.2) and a trajectory $x(t; t_0, x_0) \in R^n$ of (13.2), where t is regarded as a parameter, although their notations are the same.

We revisit three important properties for the autonomous system (13.2) as follows.

Lemma 13.1 (Translation Property) If $x(t; t_0, x_0)$ is a solution of (13.2), then so is $x(t+s; t_0, x_0)$ for any constant $s \in R$.

Proof. Since

$$x'(t+s; t_0, x_0) = \frac{d\,x(t+s; t_0, x_0)}{dt} = \frac{d\,x(t+s; t_0, x_0)}{d(t+s)} = f(x(t+s; t_0, x_0)),$$

notice that $I_{\max} = R$ for all $x_0 \in D$, so $t + s \in R$. Therefore, $x(t+s; t_0, x_0)$ is also a solution of (13.2). □

Lemma 13.2 (Uniqueness of Trajectory) If two trajectories of (13.2) have a common point, then the trajectories are identical.

Proof. Suppose that two trajectories $x(t; t_0, x_0)$ and $\tilde{x}(t; t_1, x_0)$ have a common point $x_0 \in R^n$, it must have $t_1 \neq t_0$. Otherwise, $x(t; t_0, x_0) \equiv \tilde{x}(t; t_1, x_0)$ by uniqueness of solution.

By Translation Property, it follows that $x(t+t_0-t_1; t_0, x_0)$ is also a solution of (13.2). Moreover, it has

$$x(t+t_0-t_1; t_0, x_0)|_{t=t_1} = x_0 = \tilde{x}(t; t_1, x_0)|_{t=t_1}.$$

By uniqueness of solution, it follows that $x(t+t_0-t_1; t_0, x_0) \equiv \tilde{x}(t; t_1, x_0)$. Therefore, they are identical in R^n. The uniqueness of trajectory is proved. □

Remark 13.2 Lemma 13.1 and Lemma 13.2 show that two solutions $x(t; t_0, x_0)$ and $x(t+s; t_0, x_0)$ may have different time passing through x_0, but they have the same trajectory.

Remark 13.3 Since (13.2) is time-invariant, $x(t-t_0; 0, x_0)$ is also a solution of (13.2) if $x(t; t_0, x_0)$ is a solution of (13.2). Moreover,

$$x(t-t_0; 0, x_0)|_{t=t_0} = x_0 = x(t; t_0, x_0)|_{t=t_0}.$$

Then, $x(t; t_0, x_0) \equiv x(t-t_0; 0, x_0)$ by uniqueness of solution. Therefore, taking $t_0 = 0$, without loss of generality, we denote $x(t; t_0, x_0)$ as $x(t; x_0)$. The trajectory of (13.2) is determined uniquely by its initial state x_0, independent of t_0.

Lemma 13.3 (Group Property) $x(t_1+t_2;\ x_0) = x(t_2;\ x(t_1;\ x_0))$, for $x_0 \in R^n$.

Proof. Since $x(t+t_1;\ x_0)$ and $x(t;\ x(t_1;\ x_0))$ are both solutions of (13.2), satisfying

$$x(t;\ x(t_1;\ x_0))|_{t=0} = x(t_1;\ x_0) = x(t+t_1;\ x_0)|_{t=0},$$

they are identically equal by uniqueness of solution. Taking $t = t_2$ both for

$x(t+t_1;\ x_0)$ and $x(t;\ x(t_1;\ x_0))$ results in the desired result. □

Remark 13.4 Denote $x_1 = x(t_1;\ x_0)$ and $x_2 = x(t_1 + t_2;\ x_0)$. Lemma 13.3 shows that a trajectory has an additive property for time t.

Remark 13.5 Why Lemma 13.3 is called "Group Property"?

Define transformation $x_t : x_0 \in D \to x(t; x_0) \in D$, where $t \in R$ is a parameter. Then, we have a transformation set as follows.

$$T = \{x_t \mid x_0 \to x(t; x_0),\ t \in R\}.$$

We define an addition of transformation as follows.

$$x_{t_1} + x_{t_2} =: x(t_2; x(t_1; x_0)) \quad \text{for any} \quad x_{t_1}, x_{t_2} \in T.$$

This defined addition is well defined because of Lemma13.3, satisfying:

1) Associate law: $(x_{t_1} + x_{t_2}) + x_{t_3} = x_{t_1} + (x_{t_2} + x_{t_3})$;

2) Zero Element: $x_{t=0} \in T$: $x_{t=0} = x(0; x_0) = x_0$;

3) Inverse Element: x_{-t}: $x_{-t} = x(-t; x_0)$, for any $x_t = x(t; x_0) \in T$ such that

$$x_t + x_{-t} = x(t; x(-t; x_0)) = x(t + (-t); x_0) = x(0; x_0) = x_0 = x_{t=0}.$$

Then, $T = \{x_t \mid x_0 \to x(t; x_0),\ t \in R\}$ is a group on the defined addition, which is said **a continuous transformation group on a single parameter** t. Since Lemma 13.3 plays a key role for such a group, it is also said "a group property".

Remark 13.6 Lemma 12.1-12.3 give a basic characterization of autonomous systems, which are not true for time-varying systems in general.

Definition 13.2 Let $x(t;\ x_0)$ be a solution of the system (13.2) for $[0, \infty)$. A point $x_0^0 \in D$ is an ω**-limit point** of $x(t;\ x_0)$ ($\gamma(x_0)$) if there exists $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n;\ x_0) = x_0^0$. The ω**-limit set** of $x(t;\ x_0)$ ($\gamma(x_0)$) is denoted $\Omega^+(x_0)$. A point $x_0^0 \in D$ is an α**-limit point** of $x(t;\ x_0)$ ($\gamma(x_0)$) if there exists $\{t_n\}$ with $t_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} x(t_n;\ x_0) = x_0^0$. The α**-limit set** of $x(t;\ x_0)$ ($\gamma(x_0)$) is denoted $\Omega^-(x_0)$.

Example 13.1 If $x(t;x_0)\equiv x_0$, then $\Omega^+(x_0)=\Omega^-(x_0)=\{x_0\}$. That is, the limit set of equilibrium is itself.

If $x(t;x_0)$ is a periodic orbit, then $\Omega^+(x_0)=\Omega^-(x_0)=x(t;x_0)$. That is, the limit set of a periodic orbit is also itself.

If an equilibrium $x=x_0$ is AS (unstable), then it is $\omega(\alpha)$ -limit set of its nearby trajectories; If $x(t;x_0)$ is stable (unstable) limit cycle, then $x(t;x_0)$ is $\omega(\alpha)$ -limit set of its nearby trajectories.

Lemma 13.4 If $x=x^*$ is an ω-limit point of $x(t;x_0)$, then any point on the trajectory $x(t;x^*)$ is also a ω-limit point of $x(t;x_0)$.

Proof. Since $x=x^*$ is an ω -limit point, there exists $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ s. t. $\lim_{n\to\infty}x(t_n;x_0)=x^*$ by definition. Suppose that $x(\tau;x^*)$ is any point of $x(t;x^*)$. By Lemma 13.3, we have

$$x(t_n+\tau;x_0)=x(\tau;x(t_n;x_0)).$$

Then

$$\lim_{n\to\infty}x(\tau+t_n;\ x_0)=\lim_{n\to\infty}x(\tau;x(t_n;x_0))=x(\tau;\lim_{n\to\infty}x(t_n;x_0))=x(\tau;x^*).$$

This shows that $x(\tau;x^*)$ is also a ω -limit point. □

Remark 13.6 Lemma 13.4 shows that $\Omega^+(x_0)$ consists of whole trajectories of (13.2). This is a very important property for the autonomous system.

Theorem 13.1 (LaSalle-Krosovskii's Theorem) Let $V:D\to R$ be C^1, such that (12.2a) and (12.2b) hold, i.e.

$$V(0)=0 \text{ and } V(x)>0 \text{ in } D-\{0\};$$

$$\dot{V}(x)\le 0 \text{ in } D.$$

Let $S=\{x\in R^n\,|\,\dot{V}(x)=0\}$. If there is no solution can stay identically in S, other than the trivial solution. Then, the origin of (13.2) is AS.

Proof. By (12.2a) and (12.2b), the origin is stable. We need to show only for attraction. That is, $\lim_{t\to+\infty} x(t;x_0)=0$. If we can show that $\Omega(x_0)=\{0\}$, then we will have $\lim_{t\to+\infty} x(t;x_0)=0$.

First of all, we can find $\Omega_\eta \subset B_r \subseteq D$ such that Ω_η is invariant. For any $x_0 \in \Omega_\eta$, we have $x(t;x_0)\subset\Omega_\eta$ for all $t\ge 0$. Therefore, $\Omega(x_0)\subseteq\Omega_\eta$ is non-empty because of Lemma 13.4.

Next, we show that $\Omega(x_0)=\{0\}$. By contradiction. There exists $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty} x(t_n;x_0)=x^*\neq 0$. By (12.2a) and (12.2b), $\lim_{n\to\infty} V(x(t_n;x_0))$ exists. Then, we have

$$\lim_{n\to\infty} V(x(t_n;x_0))=V(x^*)>0. \tag{13.3}$$

For $x(t;x^*)$, by (12.2a) and (12.2b), we have

$$V(x(t;x^*))\le V(x^*).$$

If $V(x(t;x^*))\equiv V(x^*)$ for all $t\ge 0$, then $\dot V(x(t;x^*))\equiv 0$, which shows that $x(t;x^*)\subseteq S$, for all $t\ge 0$. This is a contradiction to the assumption. Then, there exists $\tau>0$ s.t. $V(x(\tau;x^*))<V(x^*)$. By Lemma 13.4, $x(\tau;x^*)$ is also a ω-limit point of $x(t;x_0)$. There exists $\{\overline{t}_n\}$ with $\overline{t}_n\to\infty$ as $n\to\infty$ s. t.

$$\lim_{n\to\infty} x(\overline{t}_n;\ x_0)=x(\tau;x^*).$$

Therefore, we have

$$\lim_{n\to\infty} V(x(\overline{t}_n;\ x_0))=V(x(\tau;x^*))<V(x^*).$$

This also contradiction to (13.3). This contradiction shows that $\Omega(x_0)=\{0\}$. Therefore,

$$\overline{\lim_{t\to+\infty}}\, x(t;x_0)=\underline{\lim_{t\to+\infty}}\, x(t;x_0)=0 \quad\Rightarrow\quad \lim_{t\to+\infty} x(t;x_0)=0. \quad \square$$

Remark 13.7 LaSalle-Krosovskii's Theorem shows that there is some link among Lyapunov stability, invariance and ω-limit set. This insight is well developed by LaSalle. The more general case of this theorem is said LaSalle's Invariance principle, which will be stated next time.

Theorem 13.2 (LaSalle-Krosovskii's Theorem) Let $V : D \to R$ be a continuously differentiable function, such that (12.2a), (12.2b), and (12.3b), i.e.

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{in} \quad D - \{0\};$$

$$\dot{V}(x) \le 0 \quad \text{in} \quad D;$$

$$\| x \| \to \infty \quad \Rightarrow \quad V(x) \to \infty .$$

Let $S = \{ x \in R^n \mid \dot{V}(x) = 0 \}$. If there is no trajectory can stay identically in S, other than the origin. Then, the origin of (13.2) is GAS.

Proof. (Homework).

Example 13.2 Consider the general pendulum equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_2) \end{cases}$$

where $g(\cdot)$ and $h(\cdot)$ are locally Lip. and satisfy

$$g(0) = 0, \quad y g(y) > 0, \quad \forall\ y \ne 0, \quad y \in (-a, a);$$

$$h(0) = 0, \quad y h(y) > 0, \quad \forall\ y \ne 0, \quad y \in (-a, a).$$

If we take

$$V(x) = \int_0^{x_1} g(y)\, dy + \frac{1}{2} x_2^2 .$$

Let $D = \{ x \in R^2 \mid -a < x_j < a \}$. $V(x)$ is positive definite in D.

$$\dot{V}(x) = g(x_1) x_2 + x_2 [-g(x_1) - h(x_2)] = -x_2\, h(x_2) \le 0 .$$

Then, $S = \{ x \in D \mid \dot{V}(x) = 0 \}$, note that

$$\dot{V}(x) = 0 \quad \Rightarrow \quad x_2 h(x_2) = 0 \quad \Rightarrow \quad x_2 = 0, \text{ since } -a < x_2 < a .$$

Hence, $S = \{ x \in D \mid x_2 = 0 \}$. Suppose $x(t)$ is a trajectory that belongs identically to S. Then,

$$x_2(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_2(t) \equiv 0 \quad \Rightarrow \quad g(x_1(t)) \equiv 0 \quad \Rightarrow \quad x_1(t) \equiv 0 .$$

Therefore, the only trajectory that can stay identically in S is $x(t) \equiv 0$. Thus, the origin is AS.

Example 13.3 In Example 13.2 with $a = \infty$ and $g(\cdot)$ satisfies

$$\int_0^y g(z)dz \to \infty \quad \text{as} \quad |y| \to \infty .$$

The Lyapunov function

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

is radially unbounded. Similar to the previous example, it can be shown that $\dot{V}(x) \le 0$ in R^2, and

$$S = \{x \in R^2 \mid \dot{V}(x) = 0\} = \{x \in R^2 \mid x_2 = 0\}$$

contains no trajectory other than $x(t) \equiv 0$. Hence, the origin is GAS.

5. Region of Attraction

In real applications, local AS makes no sense. Thus for sure, we need to know the region of attraction of equilibrium, or its estimate.

Theorem 13.3 If $x = 0$ is an AS equilibrium of (12.1), then its region of attraction R_A is an open, connected, invariant set. Moreover, the boundary of R_A is formed by trajectories.

Proof. The details of proof are omitted. □

Example 13.4 For Van der Pol equation in reverse time, that is, with t replaced by $-t$, the form of the system is given

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases} .$$

There is only one equilibrium $x = 0$, which is a stable focus; hence, it is AS. This can be confirmed by linearization, since $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ has eigenvalues at $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

The system has a limit cycle, which is shown by Fig. 12.1. Clearly, the boundary of R_A is this limit cycle. Therefore, R_A is an open, connected, invariant set.

Example 13.5 Consider the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2 \end{cases} .$$

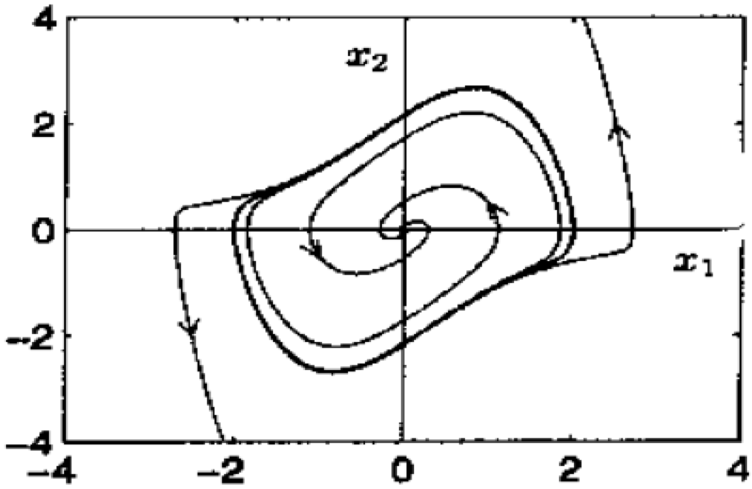

Fig. 13.1

The system has three isolated equilibriums at $(0,0)$, $(\sqrt{3},0)$ and $(-\sqrt{3},0)$. The phase portrait of the system is given by Fig.13.2. The origin is a stable focus, and the other two are saddle points. These can be confirmed by linearization.

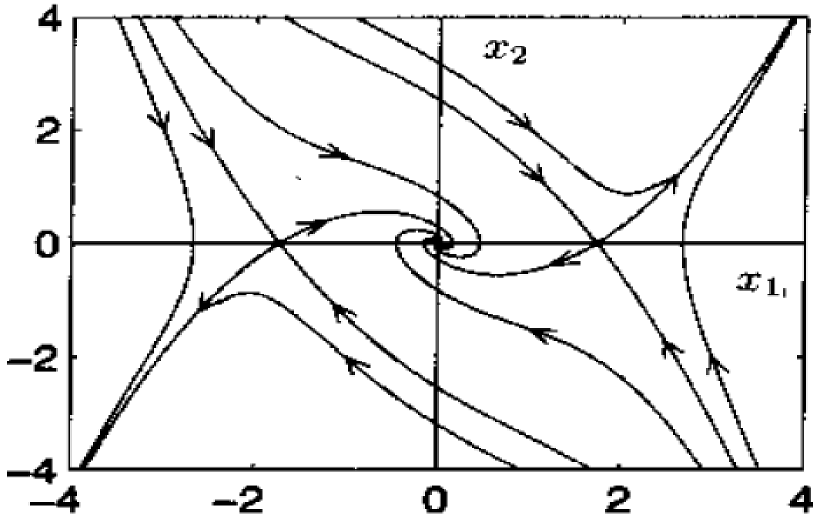

Fig. 13.2

Remark 13.8 By Theorem 13.3, finding a region of attraction is not realistic. The estimation is a direction of the work. If equilibrium is AS and $V(x)$ is a Lyapunov function, then, $\Omega_c = \{x \in R^n \,|V(x) \le c\}$ is a good estimation of a region of attraction.

Remark 13.9 For the case of $V(x) \ge 0$ and $\dot{V}(x) \le 0$, see "Semi-Definite Lyapunov Function: Stability and Stabilization" in Mathematics of Control, Signals, and

Systems (MCSS), vol. 9, no. 2, 95-106, 1996, by A. lggidr, B. Kalitine and R. Outbib, which can be download in E-Library of ECNU.

Remark 13.10 It is probably to extend LaSalle-Krosovskii's Theorem to the case of case of $V(x) \geq 0$ and $\dot{V}(x) \leq 0$ in a direct way. That is, simply taking $V(x) > 0$ to the weak condition $V(x) \geq 0$, all the other statement is the same! This conjecture is different from lggidr's work in MCSS. My personal view, yes for 90%. If no, give counterexample! You are encouraged to do it. Any students who do this work with satisfaction will be offered 100% marks in the coming final exam.

4. Summary

- LaSalle's Theorem is a fundamental result in stability analysis of nonlinear autonomous system. Any possible extensions to time-varying systems, hybrid systems, large scales systems etc. are still playing a key role and profound influence in system analysis. So its development will be interested by researchers.

- LaSalle's Theorem uses an additional condition to treat the case of $V(x) > 0$ and $\dot{V}(x) \leq 0$. This type of $V(x)$ is called a Lyapunov function. Many nonlinear systems have such a Lyapunov function. However, in controller design, a strict Lyapunov function is preferred, i.e. $V(x) > 0$ and $\dot{V}(x) < 0$. The construction of the strict Lyapunov function based on a known Lyapunov function plays a central role in nonlinear control. It receives recently much attention. Especially it is for time-varying systems. The corresponding theory is said "Strictification Method". The reference book is "Construction of Strict Lyapunov Functions" by Michael Malisoff and Frédéric Mazenc, Published by Springer, in 2009. However, it is still lots of questions unsolved and open.

Homework

1. Suppose that $V: D \to R$ be C^1, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\};$$

$$\dot{V}(x) \leq 0 \text{ in } D,$$

where D is a bounded open set in R^n. Show that if for any $c > 0$ such that the set $S = \{x \in D \,|\, V(x) \leq c\}$ is closed in R^n, then this set is positively invariant.

2. Suppose that $V : D \to R$ be C^1, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\};$$

$$\dot{V}(x) \le 0 \text{ in } D,$$

where D containing the origin is a bounded open set in R^n. Show that if $c > 0$ such that the set $S = \{x \in D \mid V(x) \le c\}$ is closed in R^n and if there is no $x \ne 0$ for which $V(\varphi(t; x))$ is constant for all $t \ge 0$, then for all $x \in S$, $\varphi(t; x) \to 0$ as $t \to \infty$, where $\varphi(t; x)$ is a solution of (13.2) satisfying the initial condition $\varphi(0; x) = x$. (A Different Version of LaSalle's Theorem)